D'Alembert's Principle

This method is based on the principle of virtual work. The system is subjected to an infinitesimal displacement consistent with the forces and constraints imposed on the system at the given instant t. This change in the configuration of the system is not associated with a change in time i.e. there is no actual displacement during which forces and constraints may change and hence the displacement is termed as virtual displacement.

Now suppose the system is in equilibrium i.e. total force $\vec{F}_i$ on every particle is zero, then work done by this force in a small virtual displacement $\delta\vec{r}_i$ will also vanish i.e. for whole system for N particles

$$\sum_{i=1}^{N} \vec{F_i} \cdot \delta \vec{r_i} = 0$$

Let this total force be expressed as sum of applied force $\vec{F_i^a}$ and forces of constraints $\vec{f_i}$. Then above equation takes the form

$$\sum_i \vec{F_i^a} \cdot \delta \vec{r_i} + \sum_i \vec{f_i} \cdot \delta \vec{r_i} = 0$$

We now consider the system for which the virtual work of the forces of constraints is zero. An example of such a system can be that of a particle which is constraind to move on a smooth surface so that the forces of constraints are perpendicular to the surface while virtual displacement is tangential to it. In such a situation, virtual work done by forces of constraints will be zero.

Thus, $$\sum_i \vec{F_i^a} \cdot \delta \vec{r_i} = 0 \qquad (1)$$

The eq^n (1) is termed as principle of virtual work. To interpret the equilibrium of the system, D'Alembert adopted an idea of a reversed force. He conceived that a system will remain in equilibrium under the action of a force equal to the actual force $\vec{F_i}$ plus reversed effective force $\dot{\vec{p}_i}$.

Thus,

$$\vec{F_i} + (-\dot{\vec{p}_i}) = 0$$

or,

$$\vec{F_i} - \dot{\vec{p}_i} = 0$$

Where, $-\dot{\vec{p}_i}$ appears as an effective force called the reversed force of inertia on the ith particle supplementing the already existing externally applied force $\vec{F_i}$.

Thus, the principle of virtual work takes the form

$$\sum_i (\vec{F_i} - \dot{\vec{p}_i}) \cdot \delta \vec{r_i} = 0$$

Again writing

$$\vec{F_i} = \vec{F_i}^a + \vec{f_i}$$

$$\sum_i (\vec{F_i}^a - \dot{\vec{p}_i}) \cdot \delta \vec{r_i} + \sum_i \vec{f_i} \cdot \delta \vec{r_i} = 0$$

Since forces of constraints are no more in picture, it is better to drop superscript a. Thus

$$\boxed{\sum_i (\vec{F_i} - \dot{\vec{p}_i}) \cdot \delta \vec{r_i} = 0} \quad \text{---} (2)$$

eq^n (2) is known as D'Alembert's Principle.
The key point is that in eq^n (2) we got rid of the forces of constraints. To satisfy eq^n (2), we can not equate the coefficients of $\delta \vec{r_i}$ to zero since $\delta \vec{r_i}$ are not independent of each other and hence it is necessary to transform $\delta \vec{r_i}$ changes into the changes of generalised co-ordinates, δq_j's which are independent of each other. The coefficient of every δq_j will then be equated to zero.